easily can be extended to investigate the dynamic stability behavior of plate and shells.

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# Zeroth-Order Shear Deformation Theory for Plates

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## Introduction

PLATE analysis involving higher-order effects such as shear is an involved and tedious process. Even the considerably simple and well-known theories such as Reissner's[1] and Mindlin's[2] require solving two differential equations involving two unknown functions and involve the use of a shear coefficient for approximately satisfying the constitutive relationship between shear stress and shear strain. However, it is possible to take into account the higher-order effects and yet keep the complexity to a considerably lower level. Great simplification is possible if axial displacement is allowed to be influenced also by shear force and if proper use is made of the relationships (which always hold, regardless of the plate theory that is used) between moments, shear forces, and loading on the plate.

In the theory to be developed, the governing differential equation is of fourth order (as is the case in classical plate theory). In the governing equation only lateral deflection, plate physical properties, and lateral loading are going to figure. Therefore, the theory developed will be called the zeroth-order shear deformation theory (ZSDT) for plates.

The essential differences between Librescu's[3] approach and the present can be summarized as follows:

Librescu's approach makes use of weighted lateral displacement, whereas the ZSDT approach uses the lateral displacement itself, and therefore the approach is physically more meaningful.

In contrast to Librescu's approach, the ZSDT approach utilizes—right from the formulation stage—only physically meaningful entities, e.g., lateral deflection and shear forces.

Note that Reissner's formulation comes out as a special case of Librescu's formulation, whereas classical plate theory comes out as a special case of ZSDT formulation. Therefore, it is the opinion of the author that, in the context of a finite element solution of thin-plate problems, if finite elements based on Librescu's approach are used, the elements will be prone to shear locking, whereas the finite elements based on the ZSDT will be free from shear locking.

## Plate Under Consideration

Consider a plate of length $a$, width $b$, and thickness $h$ of homogeneous isotropic material. In the $o - x - y - z$ Cartesian coordinate system, the plate occupies a region

$$0 \le x \le a, \qquad 0 \le y \le b, \qquad -h/2 \le z \le h/2 \tag{1}$$

The plate is loaded on surface $z = -h/2$ by a lateral load of intensity $q(x, y)$ acting in the $z$ direction. The plate can have any meaningful boundary conditions at edges $x = 0, a$ and $y = 0, b$. The modulus of elasticity $E$, shear modulus $G$, and Poisson's ratio $\mu$ are related by $G = E/[2(1+\mu)]$. The plate rigidity $D$ is defined by $D = Eh^3/[12(1-\mu^2)]$.

## Equilibrium Equations for the Plate

The moments $M_x$, $M_y$, and $M_{xy}$ and shear forces $Q_x$ and $Q_y$ are defined as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \int_{z=-h/2}^{z=h/2} \begin{Bmatrix} \sigma_x z \\ \sigma_y z \\ \tau_{xy} z \\ \tau_{zx} \\ \tau_{yz} \end{Bmatrix} \mathrm{d}z \tag{2}$$

It is worthwhile to note certain relationships between moments, shear forces, and loading:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{3}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{4}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \tag{5}$$

Equations (3–5) can be construed to be the gross equilibrium equations for any plate. As such, in the context of the classical plate theory, Eqs. (3–5) are well-known relations.[4] Note that the relations hold for any plate theory including any higher-order plate theory.

## Assumptions for ZSDT for Plates

The following are the assumptions involved in respect to ZSDT:

1) The displacements are small, and therefore the strains involved are infinitesimals.

2) The lateral displacement $w$ is a function of coordinates $x$ and $y$ only.

3) In general, stress $\sigma_z$ is negligible in comparison with $\sigma_x$ and $\sigma_y$. Therefore, for linearly elastic isotropic material it is possible to use the relations

$$\sigma_x = [E/(1-\mu^2)](\epsilon_x + \mu\epsilon_y)$$

$$\sigma_y = [E/(1-\mu^2)](\epsilon_y + \mu\epsilon_x)$$

4) The displacement $u$ in the $x$ direction and displacement $v$ in the $y$ direction each consists of two components.

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a) Linear components: The linear component of displacement $u$ and that of displacement $v$ are analogous, respectively, to the displacements $u$ and $v$ given by the classical plate theory.

b) Shear components: The shear component of displacement $u$ and that of of displacement $v$ are such that

i) they give rise to a parabolic variation of shear stresses $\tau_{zx}$ and $\tau_{yz}$ across the cross section of the plate such that the shear stresses are zero at $z = -h/2$ and at $z = h/2$.

ii) shear stresses $\tau_{zx}$ and $\tau_{yz}$ satisfy the following:

$$\int_{z=-h/2}^{z=h/2} \tau_{zx}\,\mathrm{d}z = Q_x, \qquad \int_{z=-h/2}^{z=h/2} \tau_{yz}\,\mathrm{d}z = Q_y$$

5) Body forces are assumed to be zero. (They can be treated as external forces without much loss of accuracy.)

## Displacements, Moments in ZSDT

On the basis of the assumptions made, it is easy to write

$$u = -z\frac{\partial w}{\partial x} + \frac{2(1+\mu)}{E}\left[\frac{3}{2}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]Q_x \tag{6}$$

$$v = -z\frac{\partial w}{\partial y} + \frac{2(1+\mu)}{E}\left[\frac{3}{2}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]Q_y \tag{7}$$

$$w = w(x, y) \tag{8}$$

Using displacement equations (6–8), assumptions made in respect to ZSDT for plates, and the equilibrium equations (3–5), one can obtain the expressions for the strains and stresses (strains and stresses will have linear and nonlinear components); and using these one can write expressions for $M_x$, $M_y$, $M_{xy}$:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2}{5(1-\mu)}\left(\frac{\partial Q_x}{\partial x} + \mu\frac{\partial Q_y}{\partial y}\right) \tag{9}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) + \frac{h^2}{5(1-\mu)}\left(\frac{\partial Q_y}{\partial y} + \mu\frac{\partial Q_x}{\partial x}\right) \tag{10}$$

$$M_{xy} = -D(1-\mu)\frac{\partial^2 w}{\partial x \partial y} + \frac{h^2}{10}\left(\frac{\partial Q_y}{\partial x} + \frac{\partial Q_x}{\partial y}\right) \tag{11}$$

## Governing Equations and Boundary Conditions in ZSDT

Using Eqs. (9–11) in Eqs. (3) and (4), one gets

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2}{10}\left(\frac{\partial^2 Q_x}{\partial x^2} + \frac{\partial^2 Q_x}{\partial y^2} - \frac{1+\mu}{1-\mu}\frac{\partial q}{\partial x}\right) \tag{12}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2}{10}\left(\frac{\partial^2 Q_y}{\partial x^2} + \frac{\partial^2 Q_y}{\partial y^2} - \frac{1+\mu}{1-\mu}\frac{\partial q}{\partial y}\right) \tag{13}$$

Using Eqs. (12) and (13) in Eq. (5), one gets

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left[q - \frac{h^2}{5(1-\mu)}\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2}\right)\right] \tag{14}$$

Equation (14) can be considered to be the governing equation of the plate.

Study of Eqs. (12–14) reveals that shear forces $Q_x$ and $Q_y$ are given by

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{h^2}{5(1-\mu)}\frac{\partial q}{\partial x} \tag{15}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - \frac{h^2}{5(1-\mu)}\frac{\partial q}{\partial y} \tag{16}$$

The expressions for all moments $M_x$, $M_y$, and $M_{xy}$ and shear forces $Q_x$ and $Q_y$ now have been obtained.

Now some typical boundary conditions are discussed for the edge $x = a$. Boundary conditions for other edges will follow a similar pattern.

If edge $x = a$ is simply supported, then the following conditions hold:

$$[w]_{x=a} = 0, \qquad [M_x]_{x=a} = 0$$

If edge $x = a$ is free, then the following conditions hold:

$$[M_x]_{x=a} = 0, \qquad \left[Q_x + \frac{\partial M_{xy}}{\partial y}\right]_{x=a} = 0$$

If edge $x = a$ is clamped, then two types of boundary conditions analogous to those discussed by Timoshenko and Goodier,[5] in the context of the two-dimensional theory of elasticity approach for beam analysis, are feasible. In both types, displacement $w$ is zero at the edge $x = 0$. In one type, slope $\partial w/\partial x$ is zero, whereas in another type, slope $[\partial u/\partial z]_{z=0}$ is zero at the edge; this results in specifying the slope $\partial w/\partial x$ at the edge. The boundary conditions are

EITHER $\quad [w]_{x=a} = 0, \qquad \left[\frac{\partial w}{\partial x}\right]_{x=a} = 0$

OR $\quad [w]_{x=a} = 0, \qquad \left[\frac{\partial w}{\partial x}\right]_{x=a} = \frac{3(1+\mu)}{Eh}[Q_x]_{x=a}$

## Example

Consider a plate of length $a$, width $b$, and thickness $h$ of homogeneous isotropic material. In an $o - x - y - z$ Cartesian coordinate system, the plate occupies a region defined by expressions (1). The plate has simply supported boundary conditions at edges $x = 0$, $a$ and $y = 0$, $b$. The plate is loaded on surface $z = -h/2$ by a lateral load of intensity $q(x)$ acting in the $z$ direction, given by

$$q(x) = q_o \sin(\pi x/a)\sin(\pi y/b) \tag{17}$$

Using Eqs. (14) and (17), the governing equation for the problem then is obtained as

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q_o}{D}\left[1 + \frac{h^2\pi^2}{5(1-\mu)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{18}$$

The boundary conditions for the given problem can be stated as

$$[w]_{x=0} = 0, \qquad [M_x]_{x=0} = 0, \qquad [w]_{x=a} = 0$$

$$[M_x]_{x=a} = 0, \qquad [w]_{y=0} = 0, \qquad [M_y]_{y=0} = 0$$

$$[w]_{y=b} = 0, \qquad [M_y]_{y=b} = 0$$

For the problem under consideration, it can be seen that all of the boundary conditions get satisfied if the solution is taken in the form $w = C\sin(\pi x/a)\sin(\pi y/b)$, where $C$ is an unknown constant.

Using this form in Eq. (18), one can find $C$ and then obtain the expression for $w$:

$$w = q_o \left\{ \left[ 1 + \frac{h^2\pi^2}{5(1-\mu)} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \right] \Big/ D\pi^4 \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 \right\} \times \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{19}$$

When $a = 1$, $b = 1$, $h = 0.1$, and $\mu = 0.3$, the following is observed:

The deflection $w$ at $(x = 0.5, y = 0.5)$ obtained using ZSDT $(296.0674 q_o h/E)$ differs from the exact-theory[6,7] deflection $(294.2375 q_o h/E)$ only by 0.62%, whereas use of classical plate theory gives the deflection $(280.2613 q_o h/E)$, which differs from the exact-theory deflection by $-4.75\%$.

Tensile flexural stress $\sigma_x$ at $(x = 0.5, y = 0.5, z = 0.05)$ obtained by ZSDT differs from the exact-theory stress by only 0.50%, whereas use of classical plate theory gives the stress $\sigma_x$, which differs from the corresponding exact-theory stress by $-1.43\%$.

Maximum shear stress $\tau_{zx}$ at the midpoint of edge $x = 0$, i.e., at $x = 0$, $y = 0.5$, $z = 0$, classical plate theory gives shear stress $\tau_{zx}$, which is identical up to seven significant places to that obtained by ZSDT. Exact-theory results are not available. Note that, unlike in classical plate theory, in ZSDT for plates, the constitutive relationship between shear stress and shear strain is satisfied completely.

## Conclusions

The development of a simple and easy-to-use ZSDT for plates is presented. Use of ZSDT for plates has the following advantages:

1) The governing equation is a fourth-order ordinary differential equation involving lateral deflection and loading. Lateral deflection is the only unknown function.

2) The transverse shear stresses and shear strains satisfy the constitutive relation at all points.

3) Shear stresses satisfy zero shear stress conditions at the top and bottom surfaces of the plate.

4) Unlike Reissner's[1] theory or Mindlin's[2] theory, it does not require the use of a shear coefficient.

5) The bending stresses have nonlinear components similar to that in higher-order theories.

6) The formulation is capable of dealing with two types of clamped-end conditions. (This is similar to the two types of clamped-end conditions involved in the two-dimensional theory of elasticity approach for beam analysis.)

7) Right from the formulation stage, only physically meaningful entities, e.g., lateral deflection and shear forces, are involved.

8) The efforts involved in getting the solution by ZSDT approach are only marginally greater than the efforts involved with respect to classical plate theory.

9) The classical plate theory comes out as a special case of ZSDT formulation. Therefore, in the context of a finite element solution of thin-plate problems, finite elements based on the ZSDT will be free from shear locking.

10) Results obtained are accurate. The numerical results obtained in the case of a square plate, even when the thickness-to-sideratio is 0.1, are marginally different from those obtained using exact theory.

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